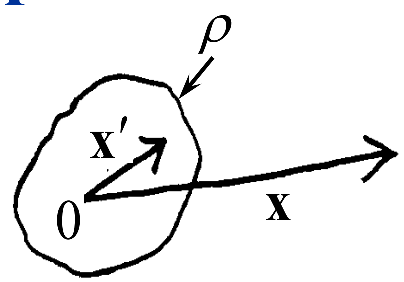
CHAPTER 4: Multipoles, Electrostatics of Macroscopic Media, Dielectrics

4.1 Multipole Expansion

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \qquad (1.17)$$

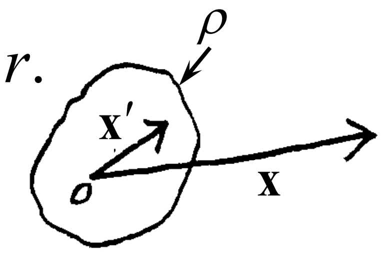


In Ch. 3, we developed various methods of expansion for the solution of the Poisson equation. In this chapter, we continue the subject of electrostatics by taking a closer look at the source $\rho(\mathbf{x})$. By the method of expansion, we first decompose $\Phi(\mathbf{x})$ in (1.17) into multipole fields and thereby express the source in multipole moments, then show that the atomic/molecular dipole moments account for the macroscopic properties of a dielectric medium and allow a concise characterization of the medium by a single number called the dielectric constant.

Multipole Expansion in Spherical Coordinates :

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \qquad (3.70)$$

For $\mathbf{x}$ outside the sphere enclosing ρ, $r_< = r'$, $r_> = r$.



$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

$$= \frac{1}{\varepsilon_0} \sum_{lm} \frac{1}{2l+1} \underbrace{\left[\int_v Y_{lm}^*(\theta',\varphi') r'^l \rho(\mathbf{x}') d^3x'\right]}_{\equiv\, q_{lm} \text{ (multipole moments)}} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \qquad (4.2)$$

monopole $(l=0)$ $[+]$ $\Rightarrow \Phi \propto \frac{1}{r}$

dipole $(l=1)$ $[+ \quad -]$ $\Rightarrow \Phi \propto \frac{1}{r^2}$

quadrupole $(l=2)$ $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ $\Rightarrow \Phi \propto \frac{1}{r^3}$

Multipole Expansion in Cartesian Coordinates :

Expansion in Cartesian coordinates is more useful for our purposes. We first summarize the formulae needed for the expansion.

Taylor expansion: [see Appendix A]

$$f(\mathbf{x}+\mathbf{a}) = f(\mathbf{x}) + (\mathbf{a}\cdot\nabla) f(\mathbf{x}) + \tfrac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) f(\mathbf{x}) + \cdots, \quad (1)$$

where

$$\mathbf{a}\cdot\nabla = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i}$$

$$(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) = \sum_i a_i \frac{\partial}{\partial x_i} \sum_j a_j \frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j \frac{\partial^2}{\partial x_i \partial x_j} \quad (2)$$

Other useful relations:

$$\nabla |\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \quad \text{[derived in Sec. 1.5]} \quad (3)$$

$$\frac{\partial}{\partial x_i} |\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(x_i - x_i') \quad (4)$$

Now apply (1)-(4) to expand $1/\left|\mathbf{x}-\mathbf{x}'\right|$.

Use (1); Write $r=\left|\mathbf{x}\right|$

$$\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}=\frac{1}{r}-\mathbf{x}'\cdot\nabla\frac{1}{r}+\frac{1}{2}(\mathbf{x}'\cdot\nabla)(\mathbf{x}'\cdot\nabla)\frac{1}{r}+\cdots$$

Use (2)-(4)

$$=\frac{1}{r}+\frac{\mathbf{x}'\cdot\mathbf{x}}{r^3}+\frac{1}{2}\sum_{ij}x_i'x_j'\underbrace{\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r}}+\cdots$$

$$=-\frac{\partial}{\partial x_i}\frac{x_j}{r^3}=-x_j\frac{\partial}{\partial x_i}\frac{1}{r^3}-\frac{1}{r^3}\underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}}=\frac{3x_ix_j}{r^5}-\frac{\delta_{ij}}{r^3}$$

$$=\sum_{ij}x_ix_j\delta_{ij}$$

$$=\frac{1}{r}+\frac{\mathbf{x}'\cdot\mathbf{x}}{r^3}+\frac{1}{2r^5}\sum_{ij}3x_i'x_j'x_ix_j-\frac{1}{2r^5}\overbrace{r^2}\overbrace{\sum_{ij}x_i'x_j'\delta_{ij}}^{r'^2}+\cdots$$

$$=\frac{1}{r}+\frac{\mathbf{x}'\cdot\mathbf{x}}{r^3}+\frac{1}{2r^5}\sum_{ij}x_ix_j\left(3x_i'x_j'-r'^2\delta_{ij}\right)+\cdots \qquad (5)$$

Multipole moments with respect to $\mathbf{x} = 0$:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

Use (5) →

$$= \frac{1}{4\pi\varepsilon_0}\Big[\overbrace{\frac{\int \rho(\mathbf{x}')d^3x'}{r}}^{q} + \frac{\mathbf{x}\cdot\overbrace{\int \mathbf{x}'\rho(\mathbf{x}')d^3x'}^{\mathbf{p}}}{r^3}$$

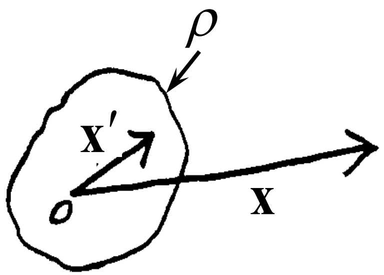
$$+ \frac{1}{2r^5}\sum_{ij} x_i x_j \overbrace{\int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')d^3x'}^{Q_{ij}} + \cdots\Big]$$

monopole moment (q)

dipole moment ($\mathbf{p}$)

quadruple moment (Q_{ij})

$$= \frac{1}{4\pi\varepsilon_0}\Big[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2r^5}\underbrace{\sum_{ij} Q_{ij}x_i x_j}_{\overleftrightarrow{\mathbf{Q}}\cdot\mathbf{xx}} + \cdots\Big]$$



Question: What is the advantage of expressing Φ this way?

$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2r^5}\overleftrightarrow{\mathbf{Q}}\cdot\mathbf{xx} + \cdots\right] \quad (4.10)$$

Note: Multipole moments are defined with repect to a point of reference. In (4.10), it is the origin of coordinates ($\mathbf{x} = 0$).

Multipole moments with respect to $\mathbf{x} = \mathbf{x}_0$:

In general, values of the multiple moments depend upon the choice of the point of reference, although the sum of all multipole fileds has the same value. Consider the general case in which the point of reference ($\mathbf{x} = \mathbf{x}_0$) is separated from the the origin of coordinates ($\mathbf{x} = 0$).

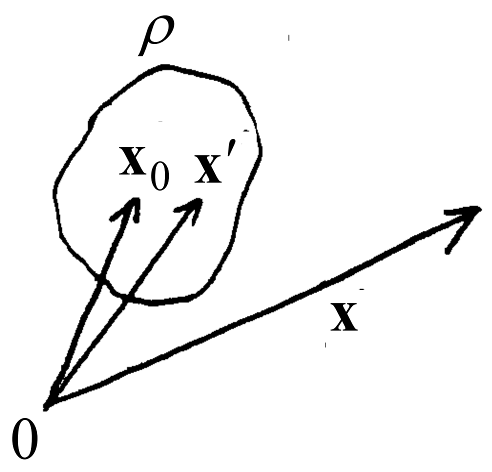
$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

$$= \frac{1}{|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|}$$

Use (5)

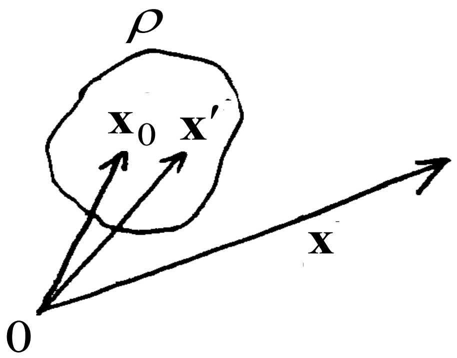
$$= \frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \frac{(\mathbf{x}'-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}$$

$$+\frac{1}{2}\sum_{ij}\frac{(x_i-x_{0i})(x_j-x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5}\left[3(x_i'-x_{0i})(x_j'-x_{0j})-|\mathbf{x}'-\mathbf{x}_0|^2\delta_{ij}\right]+\cdots$$



$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\Big[\frac{\overbrace{\int \rho(\mathbf{x}')d^3x'}^{q}}{|\mathbf{x}-\mathbf{x}_0|} + \frac{(\mathbf{x}-\mathbf{x}_0)\cdot\overbrace{\int(\mathbf{x}'-\mathbf{x}_0)\rho(\mathbf{x}')d^3x'}^{\mathbf{p}}}{|\mathbf{x}-\mathbf{x}_0|^3}$$

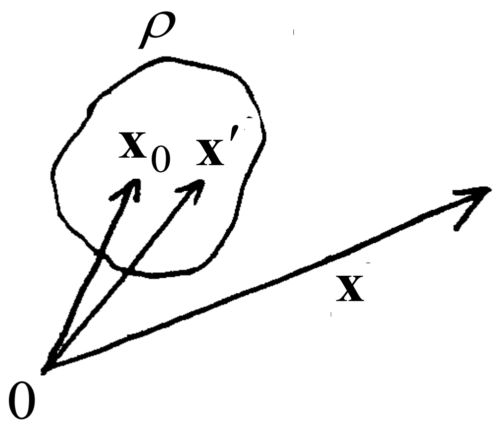


$$+\frac{1}{2}\sum_{ij}\frac{(x_i-x_{0i})(x_j-x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5}\overbrace{\int\{3(x_i'-x_{0i})(x_j'-x_{0j})-|\mathbf{x}'-\mathbf{x}_0|^2\delta_{ij}\}\rho(\mathbf{x}')d^3x'}^{Q_{ij}}+\cdots\Big]$$

$$= \frac{1}{4\pi\varepsilon_0}\Big[\underbrace{\frac{q}{|\mathbf{x}-\mathbf{x}_0|}}_{\text{due to monopole}} + \underbrace{\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}}_{\text{due to dipole}} + \underbrace{\frac{1}{2}\sum_{ij}Q_{ij}\frac{(x_i-x_{0i})(x_j-x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5}}_{\text{due to quadrupole}}+\cdots\Big] \quad (6)$$

$\mathbf{p}$ and Q_{ij} above are defined with respect to the point of reference at $\mathbf{x}_0$. We may regard $\mathbf{x}_0$ as the position of these multipoles.

Dipole field :



$$\mathbf{E}_{dipole} = -\nabla\Phi_{dipole}$$

$$= -\nabla \frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}_0|^3}$$

use (3)

$$= -\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{4\pi\varepsilon_0}\overbrace{\nabla\frac{1}{|\mathbf{x}-\mathbf{x}_0|^3}}^{\frac{-3(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^5}} - \frac{1}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_0|^3}\underbrace{\nabla[\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)]}$$

$$= \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_0|^3} \quad (4.13)$$

$$= (\mathbf{p}\cdot\nabla)(\mathbf{x}-\mathbf{x}_0) + \overbrace{[(\mathbf{x}-\mathbf{x}_0)\cdot\nabla]\mathbf{p}}^{0}$$
$$+\mathbf{p}\times\overbrace{[\nabla\times(\mathbf{x}-\mathbf{x}_0)]}^{0} + (\mathbf{x}-\mathbf{x}_0)\times\overbrace{(\nabla\times\mathbf{p})}^{0}$$
$$= (\mathbf{p}\cdot\nabla)(\mathbf{x}-\mathbf{x}_0) = \mathbf{p}$$

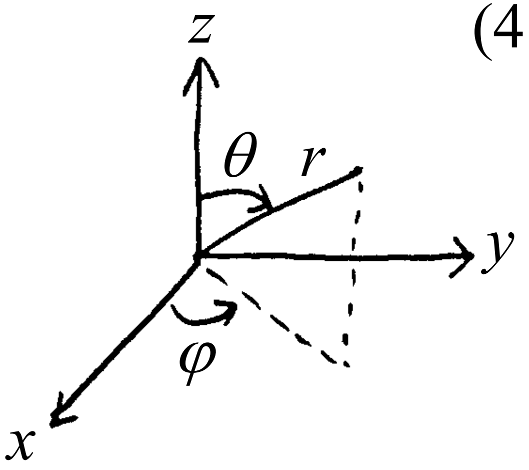
where $\mathbf{n} \equiv \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

Relation between spherical and Cartesian multipole moments :

$$\left.\begin{aligned} Y_{0,0}(\theta,\varphi) &= \sqrt{\frac{1}{4\pi}} \\ Y_{1,1}(\theta,\varphi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi} \end{aligned}\right\} \text{ p.109}$$

$$q_{lm} = \int Y_{lm}^{*}(\theta',\varphi')r'^{l}\rho(\mathrm{x}')d^{3}x' \qquad (4.3)$$



$$q_{00} = \int Y_{00}^{*}(\theta',\varphi')\rho(\mathrm{x}')d^{3}x'$$

$$= \frac{1}{\sqrt{4\pi}}\int \rho(\mathrm{x}')d^{3}x' = \frac{q}{\sqrt{4\pi}}$$

$$q_{11} = \int Y_{11}^{*}(\theta',\varphi')r'\rho(\mathrm{x}')d^{3}x'$$

$$= -\sqrt{\frac{3}{8\pi}}\int \underbrace{r'\sin\theta' e^{-i\varphi'}}\rho(\mathrm{x}')d^{3}x'$$

$r'\sin\theta'(\cos\varphi' - i\sin\varphi') = x' - iy'$

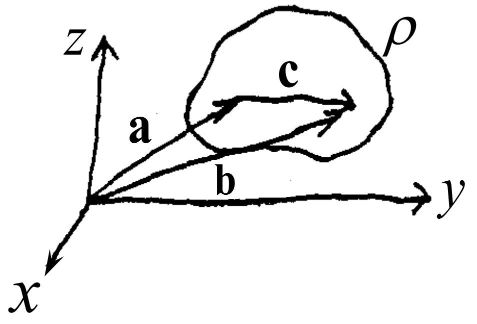
$$= -\sqrt{\frac{3}{8\pi}}(p_x - ip_y)$$

Problem: Prove that the lowest non-vanishing multipole moment is independent of the point of reference (see pp. 147-8).

Solution: Each component of the ℓ-th multipole moment with respect to reference points **a** and **b** consists, respectively, of integrals of the form $I_{ijk}^{(\mathbf{a})} = \int \rho(\mathbf{x})(x-a_x)^i (y-a_y)^j (z-a_z)^k d^3x$ and

$$I_{ijk}^{(\mathbf{b})} = \int \rho(\mathbf{x})(x-b_x)^i (y-b_y)^j (z-b_z)^k d^3x$$
$$= \int \rho(\mathbf{x})(x-a_x-c_x)^i (y-a_y-c_y)^j (z-a_z-c_z)^k d^3x,$$

where i, j, and k are zero or positive integers $(i+j+k=\ell)$, $\mathbf{a}=(a_x,a_y,a_z)$, $\mathbf{b}=(b_x,b_y,b_z)$, and $\mathbf{b}=\mathbf{a}+\mathbf{c}$ with $\mathbf{c}$ given by $\mathbf{c}=(c_x,c_y,c_z)$.

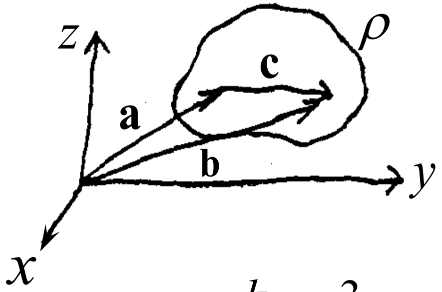


For example, the monopole moment has only one term $(i=j=k=0)$, each component of the dipole moment consists of one term (i or j or $k=1$), and each component of the quadrupole moment consists of multiple terms (all having $i+j+k=2$).

Self-learning

The monopole moment $q(=\int \rho(\mathbf{x}')d^3x')$ is clearly independent of the reference point. If $q=0$ and the lowest nonvanishing multipole moment with respect to reference point **a** is the ℓ-th moment, i.e.

$$I_{ijk}^{(\mathbf{a})} = \begin{cases} =0\ , & i+j+k<l \\ \neq 0\ , & i+j+k=l \end{cases},$$



then, with respect to reference point **b**, we have

$$\underset{i+j+k=l}{I_{ijk}^{(\mathbf{b})}} = \int \rho(\mathbf{x}) \underbrace{(x-a_x-c_x)^i}_{=(x-a_x)^i - ic_x(x-a_x)^{i-1}+..} \underbrace{(y-a_y-c_y)^j}_{=(y-a_y)^j - jc_y(y-a_y)^{j-1}+..} \underbrace{(z-a_z-c_z)^k}_{=(z-a_z)^k - kc_z(z-a_z)^{k-1}+..} d^3x$$

$$= \int_{i+j+k=l} \rho(\mathbf{x})(x-a_x)^i (y-a_y)^j (z-a_z)^k d^3x$$

$$+ \sum_{\alpha\beta\gamma} \underbrace{C_{\alpha\beta\gamma}} \underbrace{\int \rho(\mathbf{x})(x-a_x)^\alpha (y-a_y)^\beta (z-a_z)^\gamma d^3x}_{=I_{\alpha\beta\gamma}^{(\mathbf{a})}=0,\ \alpha+\beta+\gamma<l}$$

$$= \underset{i+j+k=l}{I_{ijk}^{(\mathbf{a})}}$$

multiplications of c_x, c_y, & c_z

Q.E.D.

4.2 Multipole Expansion of the Energy of a Charge Distribution in an External Field

In (1.53), we have $W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})d^3x$ [*self* energy]

potential due to $\rho(\mathbf{x})$ in the integrand, $\nabla^2\Phi(\mathbf{x}) = -\rho(\mathbf{x})/\varepsilon_0$

Here, we consider the *relative* energy between $\rho(\mathbf{x})$ and external charges:

$$W = \int \rho(\mathbf{x})\Phi(\mathbf{x})d^3x \qquad (4.21)$$

potential due to external charges, $\nabla^2\Phi(\mathbf{x}) = 0$ in region of $\rho(\mathbf{x})$

Expand the external field $\Phi(\mathbf{x})$ [Use (A.3) in appendix A]:

$$\Phi(\mathbf{x}) = \Phi(0) + \mathbf{x}\cdot\nabla\Phi(0) + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2\Phi(0)}{\partial x_i \partial x_j} + \cdots$$

$$= \Phi(0) - \mathbf{x}\cdot\mathbf{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \cdots$$

add $\underbrace{\frac{1}{6}\sum_{ij} r^2 \frac{\partial E_j(0)}{\partial x_i}\delta_{ij}}_{=\frac{1}{6}r^2\nabla\cdot\mathbf{E}(0)=0}$

$$= \Phi(0) - \mathbf{x}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j(0)}{\partial x_i} + \cdots \qquad (4.23)$$

$$\Phi(\mathbf{x}) = \Phi(0) - \mathbf{x}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j(0)}{\partial x_i} + \cdots$$

Thus,

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbf{x})d^3x$$

$$W = \int \rho(\mathbf{x})\Phi(\mathbf{x})d^3x$$

$$= q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j(0)}{\partial x_i} + \cdots \qquad (4.24)$$

$$\Rightarrow \begin{cases} q \quad [+] & \text{interacts with } \Phi \\ \mathbf{p} \quad [+ \quad -] & \text{interacts with } \mathbf{E} \text{ (non-uniform } \Phi) \\ Q_{ij} \begin{bmatrix} + & - \\ - & + \end{bmatrix} & \text{interacts with non-uniform } \mathbf{E} \end{cases}$$

Note: The multipole moments here are *not* induced by **E**. See Sec. 4.6 for induced moments.

Questions:

1. Higher order moments can “see” finer structure of $\Phi(\mathbf{x})$. Why?
2. How does a charged rod attract a piece of paper?
3. How does a microwave oven heat food?

4.6 Models for the Molecular Polarizability

KK: [mə\`lɛkjələ]

Induced Dipole Moment:

In the presence of an external electric field, the electrons and ions in a molecule (or atom) will be slightly displaced, in opposite directions, from their equilibrium positions. The molecule is thus polarized. The resulting induced dipole moment is calculated below.

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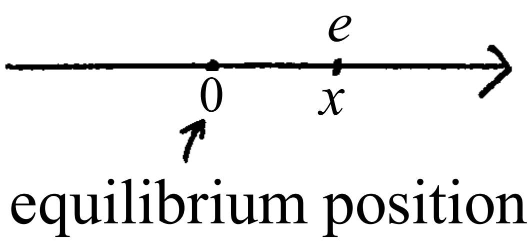
The molecular electrons and ions are bound charges. When a charge is displaced from its equilibrium position ($x = 0$), it will be subject to a restoring force $F(x)$, which we expand as

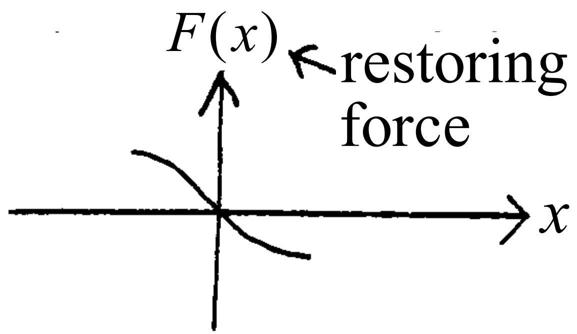


$$F(x) = \underbrace{F(0)}_{=0} + \underbrace{F'(0)}_{=-m\omega_0^2}\, x + \underbrace{\tfrac{1}{2}F''(0)x^2 + \cdots}$$

because $x = 0$ is the equilibrium position.

nonlinear effects, negligible if $x \to 0$

ω_0 is the natural frequency of the charge if it oscillates as a simple harmonic oscillator.



For small displacements ($x \to 0$) and assuming **F** and **x** are along the same line (property of an *isotropic* medium) but in opposte directions, we have

$$\mathbf{F}(\mathbf{x}) \approx -m\omega_0^2 \mathbf{x} \qquad (4.71)$$

Under the action of a static **E**, a charge will be displaced to a position **x**, at which the restoring force equals the electric force,

$$m\omega_0^2 \mathbf{x} = e\mathbf{E}$$

Note: *e* carries a sign.

This induces a dipole moment given by

$$\mathbf{p} = e\mathbf{x} = \frac{e^2}{m\omega_0^2}\mathbf{E} = \varepsilon_0 \gamma \mathbf{E}, \qquad (4.72)$$

where $\gamma \equiv e^2 / (\varepsilon_0 m\omega_0^2)$ is the polarizability of a single charge. For all the charges in the molecule, we have

$$\mathbf{p}_{mol} = \sum_j e_j \mathbf{x}_j = \sum_j \frac{e_j^2}{m_j \omega_j^2}\mathbf{E} = \varepsilon_0 \gamma_{mol} \mathbf{E},$$

[induced molecular dipole moment]

where $\gamma_{mol} \equiv \frac{1}{\varepsilon_0}\sum_j \frac{e_j^2}{m_j \omega_j^2}$ (molecular polarizability). (4.73)

Discussion:

(1) The dipole moment for a single charge as calculated above ($\mathbf{p} = e\mathbf{x}$) is with respect to the equilibrium position of the charge. Since different charges have different equilibrium positions, the dipole mements ($e_j\mathbf{x}_j$) of individual charges in the expression $\mathbf{p}_{mol} = \sum_j e_j\mathbf{x}_j$ are with respect to different reference points.

This will not cause any inconsistency for an equal amount of $+/-$ charges in the sum, in which case the monopole moment vanishes and hence $\mathbf{p}_{mol}$ is independent of the reference point (proved in Sec. 4.1). For this reason, we will assume $\mathbf{p}_{mol}$ to be contributed by an equal amount of $+/-$ charges in the molecule. If there is a net charge in the molecule, the net charge will be treated separately [see (4.29)].

(ii) The approximation made in (4.71) [$\mathbf{F}(\mathbf{x}) \approx -m\omega_0^2\mathbf{x}$] has led to a linear relation between $\mathbf{p}_{mol}$ and $\mathbf{E}$: $\mathbf{p}_{mol} = \varepsilon_0 \gamma_{mol}\mathbf{E}$.

Electric Polarization, Polarization Charge, and Free Charge:

The electric polarization is defined as the total dipole moment per unit volume and is given by

sum over all types of molecules

dipole moment per type i molecule averaged over a small volume centered at $\mathbf{x}$

$$\mathbf{P}(\mathbf{x}) = \sum_i N_i \langle \mathbf{p}_i \rangle \quad (4.28)$$

volume density of type i molecules

We now divide the charge density in a medium into two categories: polarization charge density (ρ_{pol}) and free charge density (ρ_{free}).

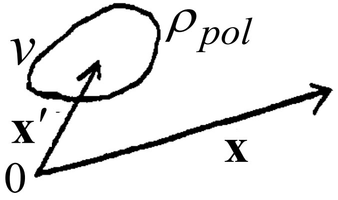
Note: We have used the notation ρ_{free} to distinguish it from ρ_{pol}. ρ_{free} here is denoted by ρ in Jackson [e.g. in (4.29), (4.35), etc.]

4.3 Elementary Treatment of Electrostatics with Ponderable Media

Macroscopic Poisson Equation : Consider a general medium and divide its charge into ρ_{free} and ρ_{pol}. By linear superposition, we may write $\Phi = \Phi_{free} + \Phi_{pol}$, where Φ_{free} and Φ_{pol} are due to ρ_{free} and ρ_{pol}, respectively. Obviously, $\Phi_{free}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$

For Φ_{pol}, we have the expression for **P**, but not yet for ρ_{pol}. So we approximate Φ_{pol} by the dipole term in (6) (with $\mathbf{x}_0$ replaced by $\mathbf{x}'$).

$$\Phi_{pol}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\cancel{\frac{q}{|\mathbf{x}-\mathbf{x}'|}} + \frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} + \cdots\right], \quad (6)$$

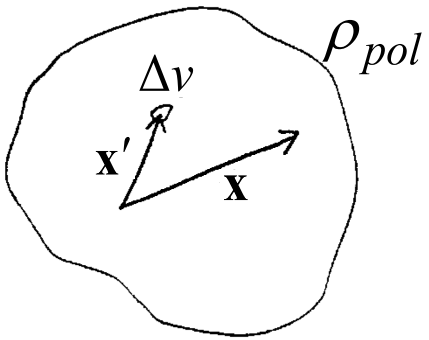


where $q = \int_v \rho_{pol} d^3x = 0$ (ρ_{pol} contains equal amount of $+/-$ charges); hence, **p** (in the volume v) is independent of the point of reference.

To represent Φ_{pol} by the dipole term in (6), we must have $|\mathbf{x}| \gg$ the dimension of **p**. So we divide ρ_{pol} into infinistesimal volumes.

Let $\Delta\Phi_{pol}(\mathbf{x})$ be the potential due to ρ_{pol} in an infinitesimal volume Δv at $\mathbf{x}'$. Then, in this volume, we have $\mathbf{p} = \mathbf{P}(\mathbf{x}')\Delta v$ and (6) gives

$$\Delta\Phi_{pol}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{P}(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')\Delta v}{|\mathbf{x}-\mathbf{x}'|^3}$$



Volume of integration includes all the charge.

$$\Rightarrow \Phi_{pol}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \underbrace{\mathbf{P}(\mathbf{x}')\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)}_{=\nabla'\cdot\frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\frac{\nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-\int d^3x' \frac{\nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \underbrace{\oint_S \frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da'}_{=0\ (\mathbf{P}=0 \text{ on } S)}\right] = -\frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Thus, $\Phi = \Phi_{free} + \Phi_{pol} = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho_{free}(\mathbf{x}')-\nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$

Question: If $\mathbf{x}-\mathbf{x}' \to 0$, higher multipole terms are important. Can we still write $\Delta\Phi_{pol}(\mathbf{x})$ and $\Phi_{pol}(\mathbf{x})$ as above?

Rewrite: $\Phi = \Phi_{free} + \Phi_{pol} = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho_{free}(\mathbf{x}') - \nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$

$$\Rightarrow \underbrace{\nabla^2\Phi(\mathbf{x})}_{-\nabla\cdot\mathbf{E}(\mathbf{x})} = \frac{1}{4\pi\varepsilon_0}\int d^3x' [\rho_{free}(\mathbf{x}') - \nabla'\cdot\mathbf{P}(\mathbf{x}')]\, \underbrace{\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}}_{-4\pi\delta(\mathbf{x}-\mathbf{x}')}$$

$$\Rightarrow \nabla\cdot\mathbf{E}(\mathbf{x}) = \frac{1}{\varepsilon_0}[\rho_{free}(\mathbf{x}) - \nabla\cdot\mathbf{P}(\mathbf{x})] \qquad (4.33)$$

In electrostatics, only the electric charge can produce **E**. The equal footing of ρ_{free} and $-\nabla\cdot\mathbf{P}$ in (4.33) suggests that $-\nabla\cdot\mathbf{P}$ (due to the electric polarization **P**) must be the polarization charge density ρ_{pol} (see p. 153 and p. 156). Thus, (4.33) can be written

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}(\rho_{free} + \rho_{pol})$$

where $$\rho_{pol} = -\nabla\cdot\mathbf{P} \qquad (7)$$

(7) is obtained here by inference. A direct derivation can be found in Appendix B [see Eq. (B.2)].

We may also put $\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{1}{\varepsilon_0}[\rho_{free}(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})]$ [(4.33)] in the form: $\nabla \cdot \mathbf{D} = \rho_{free}$ [macroscopic Poisson equation] (4.35)

by defining an electric displacement : $\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}$ (4.34)

In Sec. 4.6, by assuming an *isotropic* medium and approximating the restoring force by $\mathbf{F}(x) \approx -m\omega_0^2 \mathbf{x}$ [(4.71)], we have obtained the *linear* relation $\mathbf{p}_{mol} = \varepsilon_0 \gamma_{mol} \mathbf{E}$ for a single molecule. Then, $\mathbf{P}$ (the sum of $\mathbf{p}_{mol}$ per unit volume) must also be a linear function of $\mathbf{E}$:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad (4.36)$$

where the proportionality constant χ_e is the electric susceptibility (see Jackson Sec. 4.5 for further discussion on χ_e .). KK: [səˌsɛptəˋbɪlətɪ]

Sub. (4.36) into $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, we obtain

$$\mathbf{D} = \varepsilon \mathbf{E} \quad (4.37)$$

and $\mathbf{P} = (\varepsilon - \varepsilon_0)\, \mathbf{E}$,

where $\varepsilon = \varepsilon_0(1+\chi_e)$ (4.38)

> ε : electric permittivity
> $\varepsilon/\varepsilon_0$: dielectric constant or relative electric permittivity

Question: Is $\mathbf{D}$ a physical quantity? If so, what is its physical meaning?

Special case: For a uniform medium, ε is independent of $\mathbf{x}$. Hence, (4.35) gives $\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \varepsilon \nabla \cdot \mathbf{E} = \rho_{free}$

$$\Rightarrow \qquad \nabla \cdot \mathbf{E} = \rho_{free} / \varepsilon \quad \text{[for uniform media]} \qquad (4.39)$$

Conversion of ε to the Gaussian System:

ε in the SI system is called the electric permittivity (ε_0 is its value in vacuum). It has no counterpart in the Gaussian system. However, $\varepsilon/\varepsilon_0$ in the SI system (called dielectric constant or relative permittivity, see p.154) has a counterpart denoted by ε in the Gaussian system, According to the table on p.782, we have the following conversion formula: $\begin{bmatrix} \text{Gaussian} \\ \varepsilon \end{bmatrix} \Leftrightarrow \begin{bmatrix} \text{SI} \\ \varepsilon/\varepsilon_0 \end{bmatrix}$

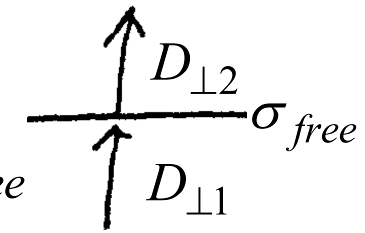
Although ε in the Gaussian system has the same notation as the electric permittivity of the SI system, it is really the dielectric constant, which corresponds to $\varepsilon/\varepsilon_0$ of the SI system. Thus, ε in these two systems are not quite the same physical quantity.

4.4 Boundary-Value Problems with Dielectrics

Boundary conditions: [σ_{free} here is denoted by σ in (4.40)]

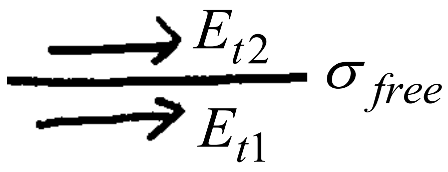
(i) $\nabla \cdot \mathbf{D} = \rho_{free}$

$\Rightarrow D_{\perp 2} - D_{\perp 1} = \sigma_{free}$

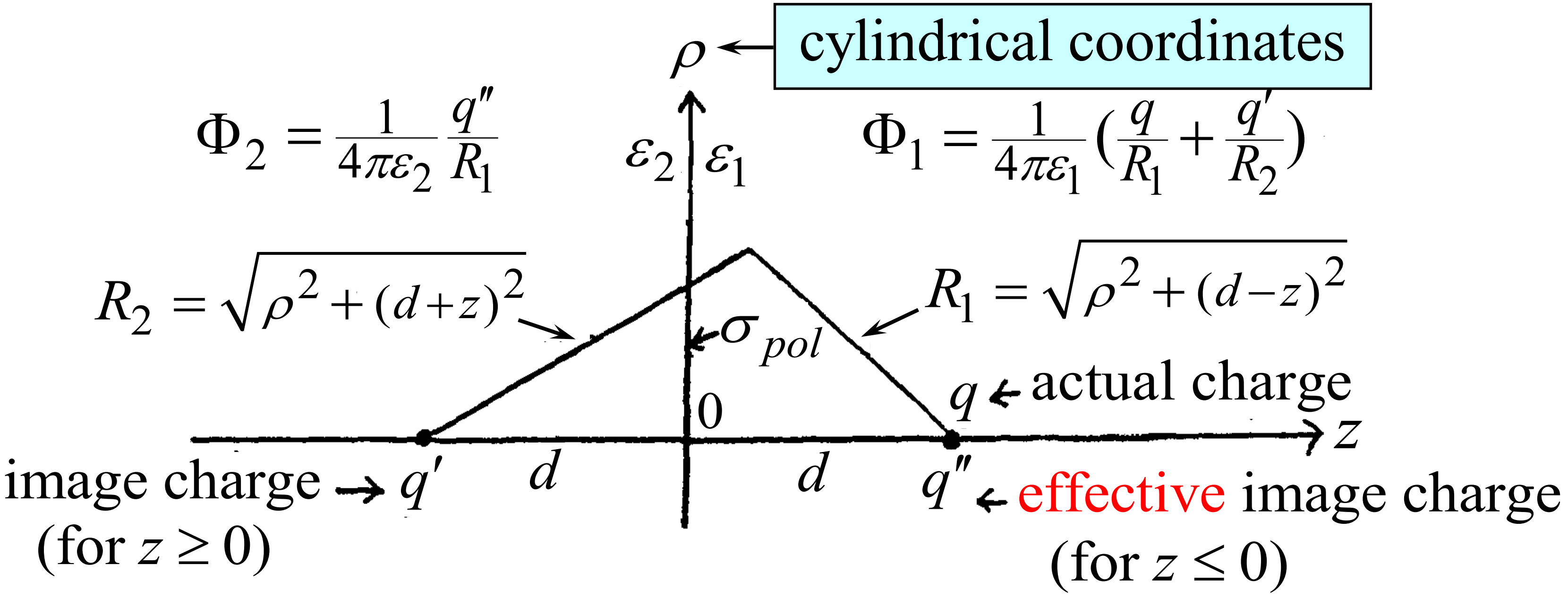


(ii) $\nabla \times \mathbf{E} = 0$

$\Rightarrow E_{t2} = E_{t1}$



Example 1: Two semi-infinite dielectrics have an interface plane at $z = 0$ (lower figure). A point charge q is at $z = d$. Find Φ everywhere.



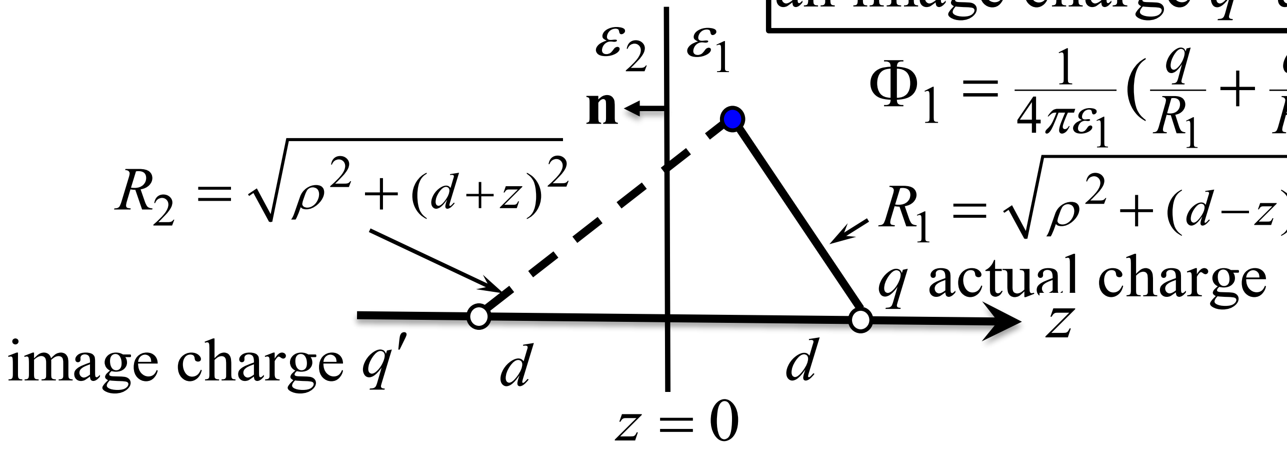
To find Φ in the region $z \geq 0$, we put an image charge q' at $z = -d$.

To find Φ in the region $z \leq 0$, we put an image charge q'' at $z = d$.

Example 1: Two semi-infinite dielectrics have an interface plane at $z=0$. A point charge q is at $z=d$. Find Φ everywhere.

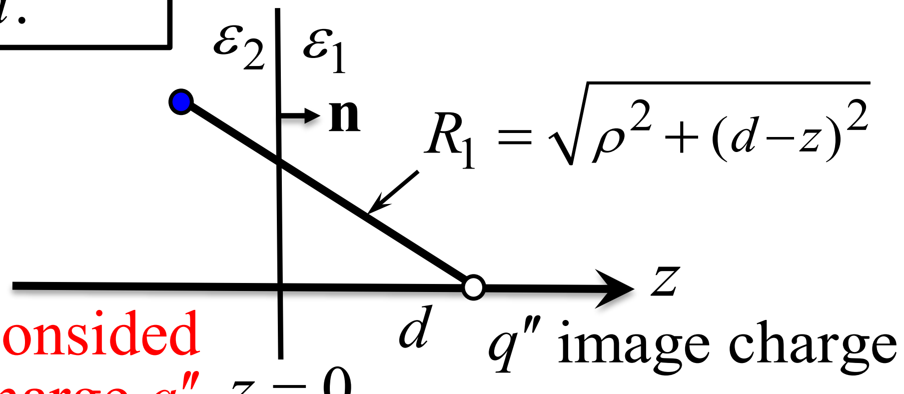
For $z \geq 0$, we put an image charge q' at $z=-d$.

$$\Phi_1 = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{R_1}+\frac{q'}{R_2}\right)$$



For $z \leq 0$, we put an effective image charge q'' at $z=d$.

$$\Phi_2 = \frac{1}{4\pi\varepsilon_2}\frac{q''}{R_1}$$



The acutal charge q is consided in the effective image charge q''.

Now apply boundary conditions at $z = 0$.

b.c. 1: $\varepsilon_1 E_{\perp 1} - \varepsilon_2 E_{\perp 2} = \sigma_{free} = 0 \Rightarrow \varepsilon_1 \left.\frac{\partial \Phi_1}{\partial z}\right|_{z=0} = \varepsilon_2 \left.\frac{\partial \Phi_2}{\partial z}\right|_{z=0}$

$$\Rightarrow \left[q \frac{\partial}{\partial z}\frac{1}{R_1} + q' \frac{\partial}{\partial z}\frac{1}{R_2} \right]_{z=0} = q'' \frac{\partial}{\partial z}\frac{1}{R_1}\bigg|_{z=0} \Rightarrow q - q' = q''$$

b.c. 2: $E_{t1} = E_{t2} \Rightarrow \left.\frac{\partial \Phi_1}{\partial \rho}\right|_{z=0} = \left.\frac{\partial \Phi_2}{\partial \rho}\right|_{z=0}$

$E_{\rho 1} = E_{\rho 2}, \; E_{\varphi 1} \overset{?}{=} E_{\varphi 2}$

$$\Rightarrow \frac{1}{\varepsilon_1}\left[q \frac{\partial}{\partial \rho}\frac{1}{R_1} + q' \frac{\partial}{\partial \rho}\frac{1}{R_2} \right]_{z=0} = \frac{1}{\varepsilon_2} q'' \frac{\partial}{\partial \rho}\frac{1}{R_1}\bigg|_{z=0} \Rightarrow \frac{1}{\varepsilon_1}(q + q') = \frac{1}{\varepsilon_2} q''$$

$$\left\{ \begin{array}{l} q - q' = q'' \\ \frac{1}{\varepsilon_1}(q + q') = \frac{1}{\varepsilon_2} q'' \end{array} \right\} \Rightarrow q' = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} q \;\; \& \;\; q'' = \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} q \quad (4.45)$$

$$\nabla \cdot \mathbf{P} = -\rho_{pol}$$

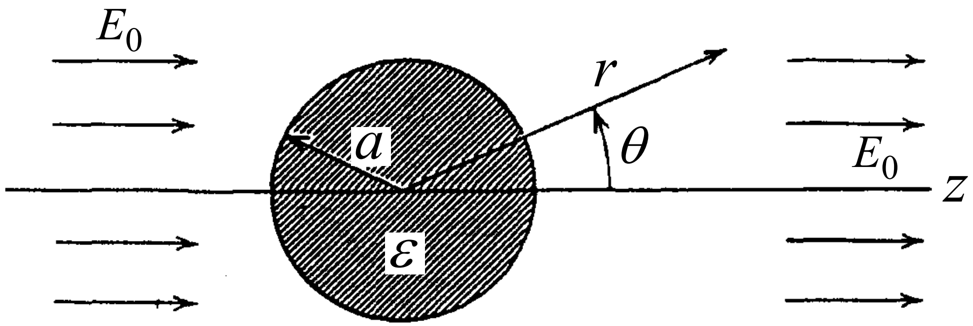
$$\Rightarrow \sigma_{pol} = -(\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n}$$

$$\mathbf{P}_1 = (\varepsilon_1 - \varepsilon_0)\mathbf{E}_1 = -(\varepsilon_1 - \varepsilon_0)\nabla \Phi_1$$
$$\mathbf{P}_2 = (\varepsilon_2 - \varepsilon_0)\mathbf{E}_2 = -(\varepsilon_2 - \varepsilon_0)\nabla \Phi_2$$

$$= -\frac{q}{2\pi} \frac{\varepsilon_0(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1(\varepsilon_2 + \varepsilon_1)} \frac{d}{(\rho^2 + d^2)^{3/2}} \quad (4.47)$$

Example 2: A dielectric sphere is placed in a uniform electric field. Find Φ everywhere.

Also see Chap. 3.3 Problem #2



We choose the spherical coordinates and divide the space into two regions: $r < a$ and $r > a$. In both regions, we have $\nabla^2\Phi = 0$ with the solution: $\Phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$ [Sec. 3.1 of lecture notes]

$$\text{b.c.} \begin{Bmatrix} \Phi \text{ is independent of } \varphi. \\ \Phi \text{ is finite at } \cos\theta = \pm 1. \\ \Phi_{in} \text{ is finite at } r = 0. \end{Bmatrix} \Rightarrow \begin{cases} \Phi_{in} = \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ \Phi_{out} = \sum\limits_{l=0}^{\infty} \left[B_l r^l + C_l r^{-l-1} \right] P_l(\cos\theta) \end{cases}$$

Question: If $l > 0$, $\Phi_{out} \to \infty$ as $r \to \infty$. Why then keep the $l > 0$ terms in Φ_{out}?

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Rewrite: $\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$, $\Phi_{out} = \sum_{l=0}^{\infty}\left[B_l r^l + C_l r^{-l-1}\right] P_l(\cos\theta)$

b.c. (i): $\Phi_{out}(\infty) = -E_0 z + const. = -E_0 r\cos\theta + const.$

$$\Rightarrow B_0 = const.;\ B_1 = -E_0;\ B_l(l>1) = 0$$

$P_1(\cos\theta) = \cos\theta$

b.c. (ii): $\Phi_{in}(a) = \Phi_{out}(a)$ [or $E_t^{in}(a) = E_t^{out}(a)$]

$$\Rightarrow A_l a^l = B_l a^l + \frac{C_l}{a^{l+1}} \Rightarrow \begin{cases} A_0 = B_0 + C_0/a, & l=0 \quad (8) \\ A_1 = -E_0 + C_1/a^3, & l=1 \quad (9) \\ A_l = C_l/a^{2l+1}, & l>1 \quad (10) \end{cases}$$

b.c. (iii): $\varepsilon E_r^{in}(a) = \varepsilon_0 E_r^{out}(a) \Rightarrow -\varepsilon \frac{\partial}{\partial r}\Phi_{in}\Big|_{r=a} = -\varepsilon_0 \frac{\partial}{\partial r}\Phi_{out}\Big|_{r=a}$

$$\Rightarrow \varepsilon l A_l a^{l-1} = \varepsilon_0\left[l B_l a^{l-1} - (l+1)C_l / a^{l+2}\right]$$

$$\Rightarrow \begin{cases} 0 = -\varepsilon_0 C_0 / a^2, & l=0 \quad (11) \\ \varepsilon A_1 = -\varepsilon_0[E_0 + 2C_1/a^3], & l=1 \quad (12) \\ \varepsilon l A_l = -\varepsilon_0 (l+1) C_l / a^{2l+1}, & l>1 \quad (13) \end{cases}$$

$$(8), (11) \Rightarrow A_0 = B_0 = const. \text{ (let it be 0.)}$$

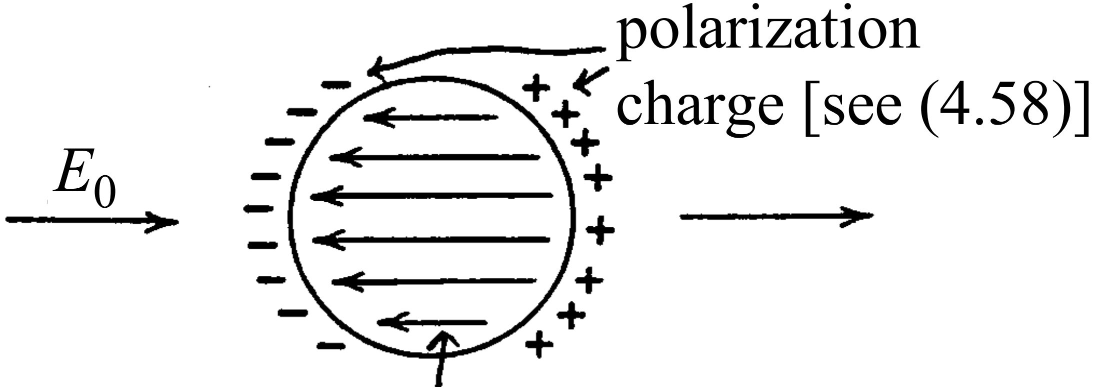
$$(9), (12) \Rightarrow A_1 = -\frac{3E_0}{2+\varepsilon/\varepsilon_0};\; C_1 = \left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)a^3E_0$$

$$(10), (13) \Rightarrow A_l = C_l = 0 \quad \text{for } l > 1$$

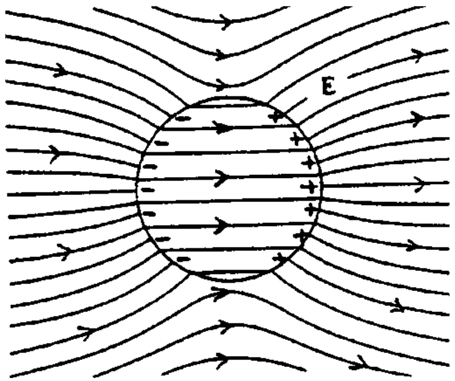
In Chap.7
$\varepsilon = \varepsilon_b + i\dfrac{\sigma}{\omega}$

$$\Rightarrow \begin{cases} \Phi_{in} = -\dfrac{3}{2+\varepsilon/\varepsilon_0}E_0 r\cos\theta \\ \Phi_{out} = \underbrace{-E_0 r\cos\theta}_{\text{applied field}} + \underbrace{\dfrac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0+2}E_0\dfrac{a^3}{r^2}\cos\theta}_{\text{dipole field with } p = 4\pi\varepsilon_0 a^3 E_0 \frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}} \end{cases} \quad (4.54)$$

[cf. (4.10)]



E due to polarization charge



total electric field

4.7 Electrostatic Energy in Dielectric Media

Let $\Phi(\mathbf{x})$ be the potential due to charge density ρ_{free}.

The work done to add $\delta\rho_{free}$ is

$$\delta W = \int \delta\rho_{free}(\mathbf{x})\Phi(\mathbf{x})d^3x$$

$\delta\rho_{free} = \nabla\cdot\delta\mathbf{D}$

$$= \int \nabla\cdot\delta\mathbf{D}(\mathbf{x})\Phi(\mathbf{x})d^3x$$

Using $\nabla\cdot\psi\mathbf{a} = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$ we obtain

$$\Phi\nabla\cdot\delta\mathbf{D} = \nabla\cdot(\Phi\delta\mathbf{D}) - \delta\mathbf{D}\cdot\nabla\Phi$$
$$= \nabla\cdot(\Phi\delta\mathbf{D}) + \mathbf{E}\cdot\delta\mathbf{D}$$

$$= \underbrace{\int \nabla\cdot(\Phi\delta\mathbf{D})d^3x}_{} + \int \mathbf{E}\cdot\delta\mathbf{D}d^3x$$

$$= \oint_S \underbrace{\Phi}_{\frac{1}{r}}\,\underbrace{\delta\mathbf{D}}_{\frac{1}{r^2}}\cdot\underbrace{d\mathbf{a}}_{r^2} = 0, \text{ as } r\to\infty$$

[For this integral to vanish, the volume of integration must be infinite.]

$$= \int \mathbf{E}\cdot\delta\mathbf{D}d^3x \qquad (4.86)$$

Note: (1) $\rho_{free}(\mathbf{x})$ here is denoted by $\rho(\mathbf{x})$ in Jackson (4.84).

(2) In a dielectric medium, the addition of $\delta\rho_{free}(\mathbf{x})$ will induce $\delta\rho_{pol}(\mathbf{x})$. Hence, $\Phi(\mathbf{x})$ in the above equation is due to both ρ_{free} and ρ_{pol}. The effect of ρ_{pol} is implicit in $\mathbf{D}$ ($=\varepsilon\mathbf{E}$).

$$\delta W = \int \mathbf{E}\cdot\delta\mathbf{D}\,d^3x \;[(4.86)] \;\Rightarrow\; W = \int d^3x \int_0^D \mathbf{E}\cdot\delta\mathbf{D} \qquad (4.87)$$

For linear and isotropic media ($\mathbf{D} = \varepsilon\mathbf{E}$; ε indep. of $\mathbf{E}$):

$$\mathbf{E}\cdot\delta\mathbf{D} = \mathbf{E}\cdot\delta(\varepsilon\mathbf{E}) = \varepsilon\mathbf{E}\cdot\delta\mathbf{E} = \tfrac{1}{2}\varepsilon\delta(\mathbf{E}\cdot\mathbf{E}) = \tfrac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D})$$

For linear and anisotropic media ($\mathbf{D} = \overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}$; $\overleftrightarrow{\boldsymbol{\varepsilon}}$ indep. of $\mathbf{E}$):

$$\mathbf{E}\cdot\delta\mathbf{D} = \mathbf{E}\cdot\delta(\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}) = \mathbf{E}\cdot\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\delta\mathbf{E} = \tfrac{1}{2}\delta(\mathbf{E}\cdot\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}) = \tfrac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D})$$

$$\Rightarrow W = \tfrac{1}{2}\int d^3x \int_0^D \delta(\mathbf{E}\cdot\mathbf{D}) = \tfrac{1}{2}\int \underbrace{\mathbf{E}\cdot\mathbf{D}}\,d^3x \quad \text{[for linear media]} \qquad (4.89)$$

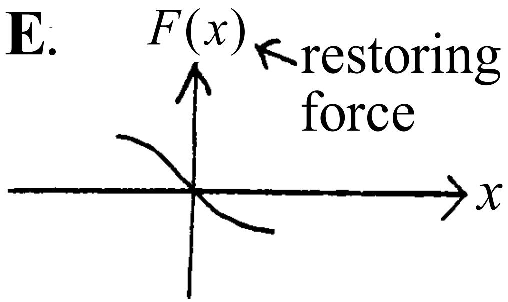
$$\mathbf{E}\cdot\mathbf{D} = -\mathbf{D}\cdot\nabla\Phi$$
$$= -\nabla\cdot(\Phi\mathbf{D}) + \Phi\nabla\cdot\mathbf{D}$$
$$= -\nabla\cdot(\Phi\mathbf{D}) + \rho_{free}\Phi$$

$$\Rightarrow W = \tfrac{1}{2}\int \rho_{free}(\mathbf{x})\Phi(\mathbf{x})d^3x - \tfrac{1}{2}\overbrace{\oint_S \underbrace{\Phi}_{\frac{1}{r}}\,\underbrace{\mathbf{D}}_{\frac{1}{r^2}}\cdot\underbrace{d\mathbf{a}}_{r^2}}^{\to 0 \text{ as } r\to\infty}$$

$$= \tfrac{1}{2}\int \rho_{free}(\mathbf{x})\Phi(\mathbf{x})d^3x \quad \text{[for linear media]}$$

Here, Φ is due to ρ_{free} and ρ_{pol}. In (1.53), $W = \tfrac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})d^3x$ (valid for a vacuum medium), Φ is due entirely to ρ in the integrand.

Example 3: Refer to the mechanism of dipole formation discussed in Sec. 4.6. Find the energy required to induce a dipole on an atomic or molecular charge e by an electric field $\mathbf{E}$.

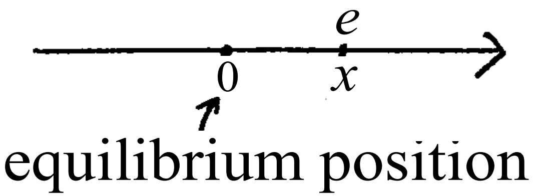


Solution: Under the restoring force:

$$\mathbf{F} = -m\omega_0^2 \mathbf{x}, \tag{4.71}$$

the energy required to displace e by a distance x is

$$w_{dipole} = -\int_0^x F(x')dx' = \frac{1}{2} m\omega_0^2 x^2$$



Using the relations:

$$\begin{cases} \text{Force balance: } m\omega_0^2 \mathbf{x} = e\mathbf{E} \Rightarrow \mathbf{E} = \dfrac{m\omega_0^2 \mathbf{x}}{e} \\ \text{Induced dipole moment: } \mathbf{p} = e\mathbf{x} \end{cases} \tag{4.72}$$

we obtain

$$w_{dipole} = \frac{1}{2} m\omega_0^2 x^2 = \frac{1}{2} \underbrace{\frac{m\omega_0^2 \mathbf{x}}{e}}_{\mathbf{E}} \cdot \underbrace{e\mathbf{x}}_{\mathbf{p}} = \frac{1}{2} \mathbf{p} \cdot \mathbf{E} \tag{14}$$

internal energy of a single dipole

Example 4: From $W = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}d^3x$ [(4.89)], we deduce that, in a dielectric, the energy density due to the presence of $\mathbf{E}$ is $w = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$. Derive this relation using the result in Example 3.

Solution: Example 3 gives the internal energy of a single dipole:

$$w_{dipole} = \frac{1}{2}\mathbf{p}\cdot\mathbf{E} \qquad (14)$$

Hence, the internal energy of all dipoles per unit volume is

$$w_{int} = \frac{1}{2}\underbrace{\sum_i N_i \mathbf{p}_i}_{\mathbf{P}} \cdot \mathbf{E} = \frac{1}{2}\mathbf{P}\cdot\mathbf{E} = \frac{1}{2}(\varepsilon - \varepsilon_0)|\mathbf{E}|^2$$

$$\begin{cases} \mathbf{D} = \varepsilon\mathbf{E} & (4.37) \\ \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} & (4.34) \end{cases} \Rightarrow \mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}$$

From Ch. 1, we have the electric field energy per unit volume:

$$w_E = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2 \qquad (1.55)$$

Hence, the total energy per unit volume is

$$w = w_{int} + w_E = \frac{1}{2}(\varepsilon - \varepsilon_0)|\mathbf{E}|^2 + \frac{1}{2}\varepsilon_0|\mathbf{E}|^2 = \frac{1}{2}\varepsilon|\mathbf{E}|^2 = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$$

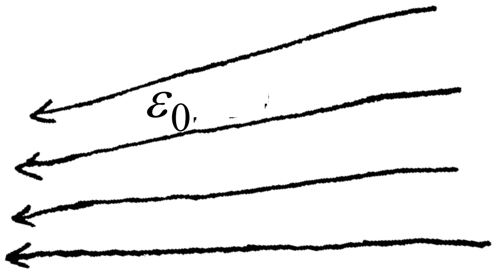
We now apply (4.89) to find the energy change due to a dielectric object with linear $\varepsilon_1(\mathbf{x})$ in the field $\mathbf{E}_0$ of a fixed external source.

Without the object:

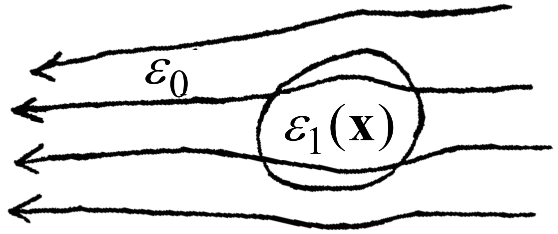
$$W_0 = \tfrac{1}{2}\int \mathbf{E}_0 \cdot \mathbf{D}_0 d^3x$$

With the object:

$$W_1 = \tfrac{1}{2}\int \mathbf{E} \cdot \mathbf{D} d^3x$$



$\mathbf{E}_0(\mathbf{x}),\ \mathbf{D}_0(\mathbf{x}),\ \Phi_0(\mathbf{x})$



$\mathbf{E}(\mathbf{x}),\ \mathbf{D}(\mathbf{x}), \Phi(\mathbf{x})$

https://www.falstad.com/vector3de/fullscreen.html

$$\Rightarrow \Delta W = W_1 - W_0 = \tfrac{1}{2}\int \left(\mathbf{E} \cdot \mathbf{D} - \mathbf{E}_0 \cdot \mathbf{D}_0\right) d^3x$$

$$= \tfrac{1}{2}\int \left(\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0\right) d^3x + \underbrace{\tfrac{1}{2}\int \left(\mathbf{E} + \mathbf{E}_0\right) \cdot \left(\mathbf{D} - \mathbf{D}_0\right) d^3x}$$

$$= \tfrac{1}{2}\int \left(\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0\right) d^3x$$

$$-\int \nabla(\Phi + \Phi_0) \cdot (\mathbf{D} - \mathbf{D}_0) d^3x = \int (\Phi + \Phi_0) \underbrace{\nabla \cdot (\mathbf{D} - \mathbf{D}_0)}_{=\rho_{free} - \rho_{free} = 0} d^3x = 0$$

integration by parts

Reason for $\nabla \cdot \mathbf{D}_0 = \nabla \cdot \mathbf{D} = \rho_{free}$: A dielectric object contains no ρ_{free} and the external source is fixed. $\Rightarrow \rho_{free}$ is unchanged before and after the introduction of the object.

$\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0$

$$\Delta W = \tfrac{1}{2}\int \left(\mathbf{E}\cdot\mathbf{D}_0 - \mathbf{D}\cdot\mathbf{E}_0\right) d^3x \qquad \begin{cases} \text{Outside the object: } \mathbf{D} = \varepsilon_0 \mathbf{E} \\ \text{Inside the object: } \mathbf{D} = \varepsilon_1 \mathbf{E} \end{cases}$$

$\Rightarrow \Delta W$ (outside the object) $= 0$

v_1 is the volume of the object.

$$\Rightarrow \Delta W = -\tfrac{1}{2}\int_{v_1} (\varepsilon_1 - \varepsilon_0)\mathbf{E}\cdot\mathbf{E}_0 d^3x \qquad (4.92)$$

$\Rightarrow$ The dielectric object tends to move toward (away from) the region of increasing $\mathbf{E}_0$ if $\varepsilon_1 > \varepsilon_0$ $(\varepsilon_1 < \varepsilon_0)$.

$\mathbf{D} = \varepsilon_1 \mathbf{E}$ & $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ $\Rightarrow \mathbf{P} = (\varepsilon_1 - \varepsilon_0)\mathbf{E}$

induced polarization of the object

$$\Rightarrow \Delta W = -\tfrac{1}{2}\int_{v_1} \mathbf{P}\cdot\mathbf{E}_0 d^3x \qquad (4.93)$$

$\Rightarrow$ The energy density of a dielectric object placed in the field $\mathbf{E}_0$ of a fixed external source is

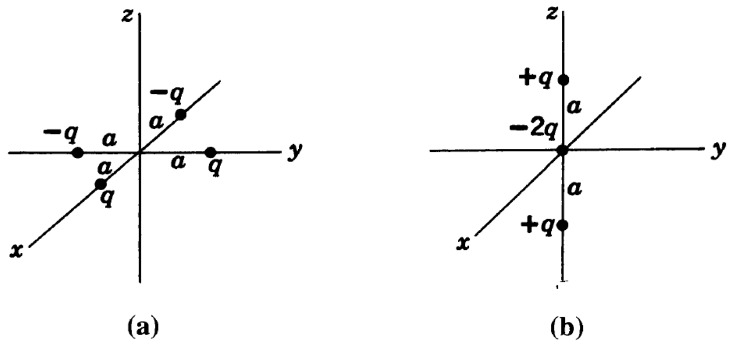
$$w = -\tfrac{1}{2}\mathbf{P}\cdot\mathbf{E}_0 \qquad (4.94)$$

Question: Explain the factor $\frac{1}{2}$ which is in (4.94) but not in the 2nd term of: $W = q\Phi(0) - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j(0)}{\partial x_i} + \cdots$ (4.24)

Homework of Chap. 4

Problem 4.1

Calculate the multipole moments q_{lm} of the charge distributions shown as parts a and b. Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.



Problem 4.1

(c) For the charge distribution of the second set b write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the $x-y$ plane as a function of distance from the origin for distances greater than a.

(d) Calculate directly from Coulomb's law the exact potential for b in the $x-y$ plane. Plot it as a function of distance and compare with the result found in part c.

Divide out the asymptotic form in parts c and d to see the behavior at large distances more clearly.

Problem 4.2

A point dipole with dipole moment $\mathbf{p}$ is located at the point $\mathbf{x}_0$, From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{eff}(x) = -\mathbf{p}\cdot\nabla\delta\left(\mathbf{x}-\mathbf{x}_0\right)$$

Homework of Chap. 4

Problem 4.7

A localized distribution of charge has a charge density

$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

(a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\mathbf{r}) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta)\right]$$

(c) If there exists at the origin a nucleus with a quadrupole moment $Q = 10^{-28}\text{m}^2$, determine the magnitude of the interaction energy, assuming that the unit of charge in $\rho(r)$ above is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\varepsilon_0\hbar^2/me^2 = 0.529\times10^{-10}\text{m}$. Express your answer as a frequency by dividing by Planck's constant h.

The charge density in this problem is that for the $m = \pm1$ states of the $2p$ level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.

Problem 4.8

A very long, right circular, cylindrical shell of dielectric constant $\varepsilon/\varepsilon_0$ and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric field in the three regions, neglecting end effects.

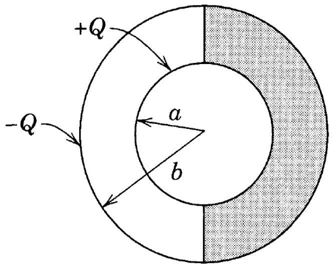
(b) Sketch the lines of force for a typical case of $b \simeq 2a$.

(c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

Homework of Chap. 4

Problem 4.10

Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\varepsilon/\varepsilon_0$), as shown in figure.



(a) Find the electric field everywhere between the spheres.

(b) Calculate the surface-charge distribution on the inner sphere.

(c) Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.

Problem 4.12

Two long, coaxial, cylinderical surfaces of radii a and b are lowered vertically into a liquid dielectric.
If the liquid rises an average height of h between the electrodes when a potential difference V is established between them, show that the susceptibility of the liquid is

$$\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\varepsilon_0 V^2}$$

where ρ is the density of the liquid of the liquid, g is the acceleration due to gravity, and the susceptibility of the air is neglected.

Appendix A. Taylor Expansion

Define $e^{\mathbf{a}\cdot\nabla} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{a}\cdot\nabla)^n$ [a translational operator, which translates the argument of the function it operates on to a distance **a** away from the argument.]

Taylor expansion of $f(\mathbf{x}+\mathbf{a})$ and $\mathbf{A}(\mathbf{x}+\mathbf{a})$ about point $\mathbf{x}$:

$$\begin{cases} f(\mathbf{x}+\mathbf{a}) = e^{\mathbf{a}\cdot\nabla} f(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{a}\cdot\nabla)^n f(\mathbf{x}) \\ \qquad = f(\mathbf{x}) + (\mathbf{a}\cdot\nabla) f(\mathbf{x}) + \frac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) f(\mathbf{x}) + \cdots \qquad (A.1) \\ \mathbf{A}(\mathbf{x}+\mathbf{a}) = e^{\mathbf{a}\cdot\nabla} \mathbf{A}(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{a}\cdot\nabla)^n \mathbf{A}(\mathbf{x}) \\ \qquad = \mathbf{A}(\mathbf{x}) + (\mathbf{a}\cdot\nabla) \mathbf{A}(\mathbf{x}) + \frac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) \mathbf{A}(\mathbf{x}) + \cdots \qquad (A.2) \end{cases}$$

Similarly, operating $f(\mathbf{x})\big|_{\text{at } \mathbf{x}=\mathbf{a}}$ and $\mathbf{A}(\mathbf{x})\big|_{\text{at } \mathbf{x}=\mathbf{a}}$ with $e^{(\mathbf{x}-\mathbf{a})\cdot\nabla}$, we obtain the Taylor expansion of $f(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ about point $\mathbf{a}$:

$$\begin{cases} f(\mathbf{x}) = f(\mathbf{a}) + [(\mathbf{x}-\mathbf{a})\cdot\nabla] f(\mathbf{a}) + \frac{1}{2}[(\mathbf{x}-\mathbf{a})\cdot\nabla][(\mathbf{x}-\mathbf{a})\cdot\nabla] f(\mathbf{a}) + \cdots \quad (A.3) \\ \mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{a}) + [(\mathbf{x}-\mathbf{a})\cdot\nabla] \mathbf{A}(\mathbf{a}) + \frac{1}{2}[(\mathbf{x}-\mathbf{a})\cdot\nabla][(\mathbf{x}-\mathbf{a})\cdot\nabla] \mathbf{A}(\mathbf{a}) + \cdots \quad (A.4) \end{cases}$$

In (A.1) and (A.2), we have [in Cartesian coordinates]

$$\mathbf{a}\cdot\nabla = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i\frac{\partial}{\partial x_i} \qquad \text{(A.5)}$$

$$(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) = \sum_i a_i\frac{\partial}{\partial x_i}\sum_j a_j\frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j\frac{\partial^2}{\partial x_i\partial x_j} \qquad \text{(A.6)}$$

$$(\mathbf{a}\cdot\nabla)f(\mathbf{x}) = \sum_i a_i\frac{\partial}{\partial x_i}f(\mathbf{x}) = \mathbf{a}\cdot\nabla f(\mathbf{x}) \qquad \text{(A.7)}$$

$$(\mathbf{a}\cdot\nabla)\mathbf{A}(\mathbf{x}) = \sum_i a_i\frac{\partial}{\partial x_i}\left(\sum_j A_j\mathbf{e}_j\right) = \sum_j\left(\sum_i a_i\frac{\partial}{\partial x_i}A_j\right)\mathbf{e}_j \qquad \text{(A.8)}$$

Example: $(\mathbf{a}\cdot\nabla)(\mathbf{x}-\mathbf{x}') = \sum_j\Big[\sum_i a_i \underbrace{\frac{\partial}{\partial x_i}(x_j - x'_j)}_{\delta_{ij}}\Big]\mathbf{e}_j = \sum_j a_j\mathbf{e}_j = \mathbf{a}$

For scalar functions with a scalar argument, (A.1) & (A.3) reduce to

$$f(x+a) = f(x) + af'(x) + \tfrac{1}{2}a^2 f''(x) + \cdots \qquad \text{(A.9)}$$

$$f(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a) + \cdots \qquad \text{(A.10)}$$

Appendix B. Polarization Current Density and Polarization Charge Density in Dielectric Media

We divide the bound charges (electrons and ions) in a dielectric into different groups. The i-th group has N_i identical charged particles per unit volume. Each particle in the group carries a charge e_i and has a dipole moment given by $\mathbf{p}_i = e_i\mathbf{x}_i$, where $\mathbf{x}_i$ is the particle's displacement from its equilibrium position under the influence of a static or time-dependent electric field. We assume that all particles in the group have the same $\mathbf{x}_i$ at all times and that the variation of $\mathbf{x}_i$ is so small that it will not change N_i. Then, the electric polarization $\mathbf{P}$ as a function of position and time can be written as

$$\mathbf{P}(\mathbf{x},t) = \sum_i N_i(\mathbf{x})\mathbf{p}_i(t) = \sum_i N_i(\mathbf{x})e_i\mathbf{x}_i(t) = \sum_i \overbrace{\rho_i(\mathbf{x})}^{\text{charge density of the } i\text{-th group}}\mathbf{x}_i(t)$$

and the polarization current density is the time derivative of $\mathbf{P}(\mathbf{x},t)$

$$\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t) = \sum_i \rho_i(\mathbf{x})\frac{d}{dt}\mathbf{x}_i(t) = \sum_i \rho_i(\mathbf{x})\mathbf{v}_i(t) = \overbrace{\mathbf{J}_{pol}(\mathbf{x},t)}^{\text{polarization current density}} \qquad \text{(B.1)}$$

Let ρ_{pol} be the polarization charge density of the medium, then

$$\frac{\partial}{\partial t}\rho_{pol} + \nabla \cdot \mathbf{J}_{pol} = 0 \qquad \text{(conservation of charge)}$$

$$\Rightarrow \frac{\partial}{\partial t}\rho_{pol} + \nabla \cdot \frac{\partial}{\partial t}\mathbf{P} = 0 \Rightarrow \frac{\partial}{\partial t}\left(\rho_{pol} + \nabla \cdot \mathbf{P}\right) = 0$$

$$\Rightarrow \rho_{pol} + \nabla \cdot \mathbf{P} = K$$

If $\mathbf{P} = 0$, we have $\rho_{pol} = 0$. Hence, $K = 0$.

$$\Rightarrow \rho_{pol} = -\nabla \cdot \mathbf{P} \qquad \text{(B.2)}$$

$\mathbf{J}_{pol}$ is due to the *motion* of bound charges, whereas ρ_{pol} is due to the *displacement* of bound charges. The presence of $\mathbf{J}_{pol}$ does not necessarily imply the presence of ρ_{pol}, and vice versa. For example, in a static electric field **E**, we have $\mathbf{J}_{pol} = 0$ because bound charges are stationary. But the stationary charges will be displaced by **E**; hence $\rho_{pol} \neq 0$ if $\nabla \cdot \mathbf{P} \neq 0$. In time-dependent cases, there must be a $\mathbf{J}_{pol}$ if $\mathbf{P} \neq 0$ [hence $\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x}, t) \neq 0$] but not necessarily a ρ_{pol} unless $\nabla \cdot \mathbf{P} \neq 0$.